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Exact Solution of 3-Dimensional Burgers Equation Using Homotopy Perturbation Method-Sumudu Transform

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Abstract

This article discusses the solution of the Burgers equation, which is a nonlinear partial differential equation, particularly for the 3D Burgers equation. This equation will be solved using a combination of the Homotopy Perturbation Method (HPM) and the Sumudu Transform (ST), known as HPM-ST. HPM-ST an alternative method to those found in the existing literature. This method is effective and easy to determining the analytic solution of nonlinear equations. To implement HPM-ST, the Sumudu transform and inverse Sumudu transform are applied first, so a nonlinear differential equation is obtained that does not depend on the variable t. Then, HPM is applied to this equation to derive an infinite series, which can be approximated using a Maclaurin series. The analytical solution of the 3D Burgers equation obtained by HPM-ST is equivalent to the exact solution. To provide an overview of the solution of the 3D Burgers equation, a visualization of the obtained solution is also presented using MATLAB.

Keywords: *3d burgers equation; homotopy perturbation method-sumudu transform; sumudu transform*

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INTRODUCTION

A differential equation is an equation that contains the relationship between the derivative of one or more unknown functions, known as dependent variables, with respect to one or more independent variables (Suryanto, 2022). Differential equations play a crucial role in various scientific fields, including mathematics, physics, chemistry, and others. One of the differential equations applied in physics is the Burgers equation. Differential equations can be classified as either linear and nonlinear. Nonlinear differential equations pose a particular challenge in obtaining analytical solutions, therefore they are often solved using numerical methods. Nevertheless, it is still possible to solve nonlinear differential equations analytically.

The Burgers equation is often used to model various physical phenomena, such as nonlinear acoustics, gas dynamics, dispersive water waves, and shock waves (Alhefthi & Eltayeb, 2023; Singh & Singh, 2022). This equation can occur in both 1D and multidimensional cases. The multidimensional Burgers equation belongs to the category of coupled Burgers equations, which is a system of equations consisting of two or more paired Burgers equations. One example of a multidimensional Burgers equation is the 3D Burgers equation. To solve the nonlinearity of the 3D Burgers equation, an appropriate method is required to obtain its analytical solution.

Various analytical methods can be used to solve nonlinear differential equations, such as the Homotopy Perturbation Method (HPM) (He, 1999), the Variational Iteration Method (VIM) (Chakraverty et al., 2019), the Laplace Variational Iteration Method (Laplace VIM) (Singh & Singh, 2022), and Homotopy Perturbation Method-Sumudu Transform (HPM-ST) (Kapoor & Joshi, 2023). Singh and Singh (2022) solved the 3D Burgers equation using a combination of the Laplace transform and the Variational Iteration Method, commonly reffered to as the Laplace VIM. The solution obtained is equivalent to the exact solution. The Laplace transform facilitates the computational process by eliminating the need for explicit integration.

Meanwhile, Watugala (1993) introduced the Sumudu transform and its applications in solving differential equations and control engineering problems. However, the Sumudu transform cannot be used to solve nonlinear problems (Kapoor, 2024). Therefore, combining the Sumudu transform with other methods is necessary to solve nonlinear differential equations. He (1999) introduced the HPM to determine analytical solutions to nonlinear equations. This method combines the Homotopy technique and the Perturbation method, which will produce solutions in the form of infinite series. Kapoor (2020) used HPM to solve the 1D coupled Burgers equation. The analytical solution of the 1D coupled Burgers can be obtained using HPM. However, in its application, integrating techniques are still needed to solve the 1D coupled Burgers equation.

Kapoor and Joshi (2023) combined HPM and the Sumudu transform to solve the 1D coupled Burgers equation and the 2D Burgers equation. This method utilizes the Sumudu transform, so obtaining a solution does not require explicit integration. Based on the results obtained, HPM-ST has proven to be effective due to its ease of implementation and its ability to produce solutions equivalent to the exact solution. HPM-ST is a combination of two effective methods to obtain analytical solutions of nonlinear equations (Sushila et al., 2014). In addition, HPM-ST has been successfully used in the Jaffery-Hamel flow equation (Sushila et al., 2014), energy balance equation (Patra & Saha Ray, 2014), 1D Keller-Segel equation (Atangana, 2015), Klein-Gordon equation (Mahdy et al., 2015), Sine-Gordon equation (Kapoor, 2022), Convection-Diffusion equation (Kapoor & Joshi, 2022), and Schrödinger equation (Kapoor, 2024).

Based on the above discussion, the application of the HPM-ST to higher-dimensional cases of the Burgers equation, such as the 3D Burgers equation, has not yet been found in the literature. Therefore, this article proposes the use of HPM-ST as an alternative approach to determine the solution of the 3D Burgers equation. The literature review discusses the governing 3D Burgers equation, the Sumudu transform, and the detailed steps of the HPM-ST. The methodology research section presents the steps for solving the 3D Burgers equation using HPM-ST. The results and discussion section presents the solution process of the 3D Burgers equation using HPM-ST, while the conclusion and suggestion section summarizes the results obtained provides recommendations for future research.

Literature Review

This section explains of the 3D Burgers equation used, the basic concept of the Sumudu transform, and the steps of the HPM-ST.

3D Burgers Equation

The 3D Burgers equation used in this article is given by equation (1) (Singh & Singh, 2022).

$$\begin{cases} \phi_t + \phi\phi_x + \psi\phi_y + \omega\phi_z = (\phi_{xx} + \phi_{yy} + \phi_{zz}), \\ \psi_t + \phi\psi_x + \psi\psi_y + \omega\psi_z = (\psi_{xx} + \psi_{yy} + \psi_{zz}), \\ \omega_t + \phi\omega_x + \psi\omega_y + \omega\omega_z = (\omega_{xx} + \omega_{yy} + \omega_{zz}), \end{cases} \tag{1}$$

with initial value in equation (2).

$$\begin{cases} \phi(x,y,z,0) = -0{,}5x + y + z, \\ \psi(x,y,z,0) = x - 0{,}5y + z, \\ \omega(x,y,z,0) = x + y - 0{,}5z, \end{cases} \tag{2}$$

and domain $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, and $z = 0{,}5$. The exact solution of equation (1) is represented by equation (3).

$$\begin{cases} \phi(x,y,z,t) = \frac{(-0{,}5x+y+z)-2{,}25xt}{1-2{,}25t^2}, \text{ with the condition } 2{,}25t^2 \neq 1, \\ \psi(x,y,z,t) = \frac{(x-0{,}5y+z)-2{,}25yt}{1-2{,}25t^2}, \text{ with the condition } 2{,}25t^2 \neq 1, \\ \omega(x,y,z,t) = \frac{(x+y-0{,}5z)-2{,}25zt}{1-2{,}25t^2}, \text{ with the condition } 2{,}25t^2 \neq 1. \end{cases} \tag{3}$$

Sumudu Transform

Definition 1. (Watugala, 1993) For every real number $t > 0$, the Sumudu transform $F(\theta)$ of a function $f(t)$ is defined as follows:

$$\boldsymbol{F(\theta) = S[f(t)] = \frac{1}{\theta}\int_0^\infty e^{-\frac{t}{\theta}} f(t)\, dt.}$$

For every real number $t > 0$, the Sumudu transform of some functions $f(t)$ is shown in Table 1.

Table 1. Sumudu transform of $f(t)$

No	$f(t)$	$F(\theta)$
1.	1	1
2.	t	θ
3.	$\frac{t^n}{n!}, n = 0, 1, 2, 3, \ldots$	θ^n
4.	e^{at}	$\frac{1}{1 - a\theta}$
5.	$\frac{\sin(at)}{a}$	$\frac{\theta}{1 + a^2\theta^2}$

No	$f(t)$	$F(\theta)$
6.	$\cos(at)$	$\dfrac{1}{1+a^2\theta^2}$
7.	$\dfrac{\sinh(at)}{a}$	$\dfrac{\theta}{1-a^2\theta^2}$
8.	$\cosh(at)$	$\dfrac{1}{1-a^2\theta^2}$

Definition 2. (Moltot & Deresse, 2022) The inverse Sumudu transform of a function $F(\theta)$ represented as $S^{-1}[F(\theta)]$ is defined as follows:

$$S^{-1}[F(\theta)] = f(t).$$

Theorem 1. (Moltot & Deresse, 2022) If $F(\theta)$ is the Sumudu transform of $f(t)$, then the Sumudu transforms of the derived functions $f'(t)$, $f''(t)$, and $f^{(n)}(t)$ are given by:

$$S[f'(t)] = \frac{F(\theta) - f(0)}{\theta} = \frac{F(\theta)}{\theta} - \frac{f(0)}{\theta},$$
$$S[f''(t)] = \frac{F(\theta)}{\theta^2} - \frac{f(0)}{\theta^2} - \frac{f'(0)}{\theta},$$
$$S\left[f^{(n)}(t)\right] = \frac{F(\theta)}{\theta^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\theta^{n-k}}.$$

Theorem 2. (Moltot & Deresse, 2022) Suppose $f(t)$ and $g(t)$ are two functions that have the Sumudu transform. For arbitrary constants a and b, the following property holds:

$$S[af(t) + bg(t)] = aS[f(t)] + bS[g(t)].$$

Theorem 3. (Moltot & Deresse, 2022) If $S[f(t)] = F(\theta)$, then the Sumudu transform of $f(at)$ is given by:

$$S[f(at)] = F(a\theta),$$

with a being a non-zero constant.

Theorem 4. (Moltot & Deresse, 2022) If $S[f(t)] = F(\theta)$, then:

$$S[f(t)e^{at}] = \frac{1}{1 - a\theta} F\left(\frac{\theta}{1 - a\theta}\right).$$

Homotopy Perturbation Method-Sumudu Transform (HPM-ST)

HPM-ST is a combination of the Sumudu transform, HPM, and He's polynomial (Patra & Saha Ray, 2014). To illustrate HPM-ST, consider the following nonhomogeneous partial differential equation (Kapoor & Joshi, 2023; Patra & Saha Ray, 2014; Sushila et al., 2014):

$$R(\phi) + L(\phi) + N(\phi) = g(x, t), \tag{4}$$

where $\phi \equiv \phi(x,t)$. R is a differential operator of order n with respect to variable t or $\frac{\partial^n}{\partial t^n}$, L is a linear differential operator, N is a nonlinear differential operator, and $g(x,t)$ is a source term. By applying the Sumudu transformation to both sides of equation (4), we obtain

$$S[\phi] = \theta^n \sum_{k=0}^{n-1} \frac{\phi^{(k)}(0)}{\theta^{n-k}} + \theta^n g(x,t) - \theta^n S[L(\phi) + N(\phi)]. \tag{5}$$

Next, the inverse Sumudu transform is applied to both sides of equation (5), thus obtaining

$$\phi = G(x,t) - S^{-1}\left[\theta^n S[L(\phi) + N(\phi)]\right], \tag{6}$$

with $G(x,t) = S^{-1}\left[\theta^n \sum_{k=0}^{n-1} \frac{\phi^{(k)}(0)}{\theta^{n-k}} + \theta^n g(x,t)\right]$. Next, HPM is applied to equation (6). Equation (6) is converted into homotopy form, so that the following equation is obtained

$$\Phi = G(x,t) - p\left(S^{-1}\left[\theta^n S[L(\Phi) + N(\Phi)]\right]\right), \tag{7}$$

with $p \in [0,1]$ being the homotopy parameter and $\Phi \equiv \Phi(x,t,p)$. The homotopy parameter p is also applied to Φ, so we obtained

$$\Phi = \sum_{m=0}^{\infty} p^m \Phi_m, \tag{8}$$

and its nonlinear terms can be written as

$$N(\Phi) = \sum_{m=0}^{\infty} p^m H_m. \tag{9}$$

The H_m is a He's polynomial of the form

$$H_m(\Phi_0, \cdots, \Phi_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[N\left(\sum_{m=0}^{\infty} p^m \Phi_m\right)\right],$$

with $m = 0, 1, 2, 3, \cdots$ and $\Phi_m \equiv \Phi_m(x,y,z,t)$. Equations (8) and (9) are substituted into equation (7), thus obtained

$$\sum_{m=0}^{\infty} p^m \Phi_m = G(x,t) - p\left(S^{-1}\left[\theta^n S\left[L\left(\sum_{m=0}^{\infty} p^m \Phi_m\right) + \sum_{m=0}^{\infty} p^m H_m\right]\right]\right). \tag{10}$$

Next, by comparing the power coefficient p of both sides of equation (10), we obtain

$$p^0: \Phi_0 = G(x,t),$$
$$p^1: \Phi_1 = -S^{-1}\left[\theta^n S[L(\Phi_0) + N(\Phi_0)]\right],$$
$$p^2: \Phi_2 = -S^{-1}\left[\theta^n S[L(\Phi_1) + N(\Phi_1)]\right],$$
$$p^3: \Phi_3 = -S^{-1}\left[\theta^n S[L(\Phi_2) + N(\Phi_2)]\right].$$

The other components Φ_m can be obtained in the same way, thus forming a series solution

$$\Phi = \lim_{N\to\infty} \sum_{m=0}^{N} p^m \Phi_m.$$

Thus, the approximate solution is obtained as follows

$$\phi(x,t) = \lim_{p\to 1} \Phi.$$

METHODS

The procedure to obtain the exact solution of the three-dimensional Burgers equation involves several analytical steps. Initially, the Sumudu transform was applied to equation (1), specifically to the derivatives of the functions $\phi(x,y,z,t)$, $\psi(x,y,z,t)$, and $\omega(x,y,z,t)$, in order to simplify the system by eliminating the time variable. Subsequently, the inverse Sumudu transform was employed on the resulting expressions, so that a nonlinear partial differential equation independent of the variable t is obtained.

The next step involved the application of the Homotopy Perturbation Method (HPM) as follows:

a. The equation obtained from the inverse Sumudu transform is transformed into the homotopy form by applying the homotopy parameter $p \in [0,1]$.
b. Applying the homotopy parameter to Φ, Ψ, Ω, and the nonlinear terms. In this step, He's polynomials are used to represent the nonlinear components.
c. Substituting the expressions obtained in steps b into the equation formed in step a.
d. Comparing the coefficients of like powers of p to obtain the terms $\Phi_0(x,y,z,t), \Phi_1(x,y,z,t), \Phi_2(x,y,z,t)$, and so on, as well as $\Psi_0(x,y,z,t), \Psi_1(x,y,z,t), \Psi_2(x,y,z,t)$, and so on, and similarly $\Omega_0(x,y,z,t), \Omega_1(x,y,z,t), \Omega_2(x,y,z,t)$, and so on.
e. Approximating the series solution to obtain $\phi(x,y,z,t)$, $\psi(x,y,z,t)$, and $\omega(x,y,z,t)$.

Finally, the resulting solutions were visualized using MATLAB at selected time values, specifically are $t = 0$ and $t = 0.6$, to illustrate the behavior of the system over time.

RESULTS AND DISCUSSION

Consider that equation (1) can be expressed as equation (11).

$$\begin{cases} \phi_t = \phi_{xx} + \phi_{yy} + \phi_{zz} - \phi\phi_x - \psi\phi_y - \omega\phi_z, \\ \psi_t = \psi_{xx} + \psi_{yy} + \psi_{zz} - \phi\psi_x - \psi\psi_y - \omega\psi_z, \\ \omega_t = \omega_{xx} + \omega_{yy} + \omega_{zz} - \phi\omega_x - \psi\omega_y - \omega\omega_z, \end{cases} \tag{11}$$

with initial value

$$\begin{cases} \phi(x,y,z,0) = -0{,}5x + y + z, \\ \psi(x,y,z,0) = x - 0{,}5y + z, \\ \omega(x,y,z,0) = x + y - 0{,}5z. \end{cases}$$

By applying Sumudu transform to both sides of equation (11), we obtain $S[\phi(x,y,z,t)]$ which can be seen in equation (12) as follows

$$\begin{aligned} S[\phi_t] &= S\left[\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi\phi_x - \psi\phi_y - \omega\phi_z\right] \\ \frac{1}{\theta}\left(S[\phi(x,y,z,t)] - \phi(x,y,z,0)\right) &= S\left[\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi\phi_x - \psi\phi_y - \omega\phi_z\right] \\ S[\phi(x,y,z,t)] &= \phi(x,y,z,0) + \theta S\left[\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi\phi_x \right. \\ &\quad \left. -\psi\phi_y - \omega\phi_z\right]. \end{aligned} \tag{12}$$

Next, we obtain $S[\psi(x,y,z,t)]$ in equation (13).

$$\begin{aligned} S[\psi_t] &= S\left[\psi_{xx} + \psi_{yy} + \psi_{zz} - \phi\psi_x - \psi\psi_y - \omega\psi_z\right] \\ \frac{1}{\theta}\left(S[\psi(x,y,z,t)] - \psi(x,y,z,0)\right) &= S\left[\psi_{xx} + \psi_{yy} + \psi_{zz} - \phi\psi_x - \psi\psi_y - \omega\psi_z\right] \\ S[\psi(x,y,z,t)] &= \psi(x,y,z,0) + \theta S\left[\psi_{xx} + \psi_{yy} + \psi_{zz} - \phi\psi_x \right. \\ &\quad \left. -\psi\psi_y - \omega\psi_z\right], \end{aligned} \tag{13}$$

and $S[\omega(x,y,z,t)]$ is obtained as shown in equation (14).

$$\begin{aligned} S[\omega_t] &= S\left[\omega_{xx} + \omega_{yy} + \omega_{zz} - \phi\omega_x - \psi\omega_y - \omega\omega_z\right] \\ \frac{1}{\theta}\left(S[\omega(x,y,z,t)] - \omega(x,y,z,0)\right) &= S\left[\omega_{xx} + \omega_{yy} + \omega_{zz} - \phi\omega_x - \psi\omega_y - \omega\omega_z\right] \\ S[\omega(x,y,z,t)] &= \omega(x,y,z,0) + \theta S\left[\omega_{xx} + \omega_{yy} + \omega_{zz} - \phi\omega_x \right. \\ &\quad \left. -\psi\omega_y - \omega\omega_z\right]. \end{aligned} \tag{14}$$

The initial value is substituted into equations (12), (13), and (14), resulting in the following equation (15).

$$\begin{cases} S[\phi(x,y,z,t)] = (-0{,}5x + y + z) + \theta S\left[\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi\phi_x - \psi\phi_y - \omega\phi_z\right], \\ S[\psi(x,y,z,t)] = (x - 0{,}5y + z) + \theta S\left[\psi_{xx} + \psi_{yy} + \psi_{zz} - \phi\psi_x - \psi\psi_y - \omega\psi_z\right], \\ S[\omega(x,y,z,t)] = (x + y - 0{,}5z) + \theta S\left[\omega_{xx} + \omega_{yy} + \omega_{zz} - \phi\omega_x - \psi\omega_y - \omega\omega_z\right]. \end{cases} \tag{15}$$

By applying the inverse Sumudu transform to both sides of equation (15), we obtain the following equation (16).

$$\begin{cases} \phi(x,y,z,t) = (-0{,}5x + y + z) + S^{-1}\left[\theta S\left[\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi\phi_x - \psi\phi_y - \omega\phi_z\right]\right], \\ \psi(x,y,z,t) = (x - 0{,}5y + z) + S^{-1}\left[\theta S\left[\psi_{xx} + \psi_{yy} + \psi_{zz} - \phi\psi_x - \psi\psi_y - \omega\psi_z\right]\right], \\ \omega(x,y,z,t) = (x + y - 0{,}5z) + S^{-1}\left[\theta S\left[\omega_{xx} + \omega_{yy} + \omega_{zz} - \phi\omega_x - \psi\omega_y - \omega\omega_z\right]\right]. \end{cases} \tag{16}$$

Then, HPM is applied to equation (16). Equation (16) is converted into homotopy form, thus yielding the following equation (17).

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$$\begin{cases} \Phi = (-0{,}5x + y + z) + p\left(S^{-1}\left[\theta S\left[\Phi_{xx} + \Phi_{yy} + \Phi_{zz} - \Phi\Phi_x - \Psi\Phi_y - \Omega\Phi_z\right]\right]\right), \\ \Psi = (x - 0{,}5y + z) + p\left(S^{-1}\left[\theta S\left[\Psi_{xx} + \Psi_{yy} + \Psi_{zz} - \Phi\Psi_x - \Psi\Psi_y - \Omega\Psi_z\right]\right]\right), \\ \Omega = (x + y - 0{,}5z) + p\left(S^{-1}\left[\theta S\left[\Omega_{xx} + \Omega_{yy} + \Omega_{zz} - \Phi\Omega_x - \Psi\Omega_y - \Omega\Omega_z\right]\right]\right), \end{cases} \tag{17}$$

with $p \in [0,1]$, $\Phi \equiv \Phi(x, y, z, t, p)$, $\Psi \equiv \Psi(x, y, z, t, p)$, and $\Omega \equiv \Omega(x, y, z, t, p)$. Φ, Ψ, and Ω can be written in the form of equations (18), (19), and (20) as follows

$$\Phi = \sum_{m=0}^{\infty} p^m \Phi_m, \tag{18}$$

$$\Psi = \sum_{m=0}^{\infty} p^m \Psi_m, \tag{19}$$

$$\Omega = \sum_{m=0}^{\infty} p^m \Omega_m, \tag{20}$$

with $\Phi_m \equiv \Phi_m(x, y, z, t)$, $\Psi_m \equiv \Psi_m(x, y, z, t)$, and $\Omega_m \equiv \Omega_m(x, y, z, t)$. The nonlinear terms are expressed as $H(\Phi, \Psi, \Omega) = \Phi\Phi_x + \Psi\Phi_y + \Omega\Phi_z$, $I(\Phi, \Psi, \Omega) = \Phi\Psi_x + \Psi\Psi_y + \Omega\Psi_z$, and $J(\Phi, \Psi, \Omega) = \Phi\Omega_x + \Psi\Omega_y + \Omega\Omega_z$, which can be expanded into equations (21), (22), and (23).

$$H(\Phi, \Psi, \Omega) = \sum_{m=0}^{\infty} p^m H_m, \tag{21}$$

$$I(\Phi, \Psi, \Omega) = \sum_{m=0}^{\infty} p^m I_m, \tag{22}$$

$$J(\Phi, \Psi, \Omega) = \sum_{m=0}^{\infty} p^m J_m. \tag{23}$$

H_m, I_m, and J_m are He's polynomial with the following details

$$H_m = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[H\left(\sum_{i=0}^{\infty} p^i \Phi_i, \sum_{i=0}^{\infty} p^i \Psi_i, \sum_{i=0}^{\infty} p^i \Omega_i\right)\right]_{p=0},$$

$$I_m = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[I\left(\sum_{i=0}^{\infty} p^i \Phi_i, \sum_{i=0}^{\infty} p^i \Psi_i, \sum_{i=0}^{\infty} p^i \Omega_i\right)\right]_{p=0},$$

$$J_m = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[J\left(\sum_{i=0}^{\infty} p^i\Phi_i, \sum_{i=0}^{\infty} p^i\Psi_i, \sum_{i=0}^{\infty} p^i\Omega_i\right)\right]_{p=0}.$$

Equations (18) through (23) are substituted into equation (17), obtaining equation (24) through equation (26) as follows.

$$\sum_{m=0}^{\infty} p^m\Phi_m = (-0{,}5x + y + z) + p\left(S^{-1}\left[\theta S\left[\left(\sum_{m=0}^{\infty} p^m\Phi_m\right)_{xx} + \left(\sum_{m=0}^{\infty} p^m\Phi_m\right)_{yy} + \left(\sum_{m=0}^{\infty} p^m\Phi_m\right)_{zz} - \sum_{m=0}^{\infty} p^m H_m\right]\right]\right), \tag{24}$$

$$\sum_{m=0}^{\infty} p^m\Psi_m = (x - 0{,}5y + z) + p\left(S^{-1}\left[\theta S\left[\left(\sum_{m=0}^{\infty} p^m\Psi_m\right)_{xx} + \left(\sum_{m=0}^{\infty} p^m\Psi_m\right)_{yy} + \left(\sum_{m=0}^{\infty} p^m\Psi_m\right)_{zz} - \sum_{m=0}^{\infty} p^m I_m\right]\right]\right), \tag{25}$$

$$\sum_{m=0}^{\infty} p^m\Omega_m = (x + y - 0{,}5z) + p\left(S^{-1}\left[\theta S\left[\left(\sum_{m=0}^{\infty} p^m\Omega_m\right)_{xx} + \left(\sum_{m=0}^{\infty} p^m\Omega_m\right)_{yy} + \left(\sum_{m=0}^{\infty} p^m\Omega_m\right)_{zz} - \sum_{m=0}^{\infty} p^m J_m\right]\right]\right). \tag{26}$$

In equations (24) to (26), the polynomial on the left side will be the same as the polynomial on the right side if each p^m term has the same coefficient. Therefore, by comparing the same terms on both sides of equations (24) to (26), we obtain:

The p^0 term produces Φ_0, Ψ_0, and Ω_0 as follows

$$\begin{cases} \Phi_0 = -0{,}5x + y + z, \\ \Psi_0 = x - 0{,}5y + z, \\ \Omega_0 = x + y - 0{,}5z. \end{cases}$$

Next, the p^1 term is obtained as shown in equation (27).

$$\begin{cases} \Phi_1 = S^{-1}\left[\theta S\left[(\Phi_0)_{xx} + (\Phi_0)_{yy} + (\Phi_0)_{zz} - H_0\right]\right], \\ \Psi_1 = S^{-1}\left[\theta S\left[(\Psi_0)_{xx} + (\Psi_0)_{yy} + (\Psi_0)_{zz} - I_0\right]\right], \\ \Omega_1 = S^{-1}\left[\theta S\left[(\Omega_0)_{xx} + (\Omega_0)_{yy} + (\Omega_0)_{zz} - J_0\right]\right]. \end{cases} \tag{27}$$

The values of H_0, I_0, and J_0 are given by equations (28), (29), dan (30).

$$H_0 = \Phi_0(\Phi_0)_x + \Psi_0(\Phi_0)_y + \Omega_0(\Phi_0)_z = 2{,}25x, \tag{28}$$

$$I_0 = \Phi_0(\Psi_0)_x + \Psi_0(\Psi_0)_y + \Omega_0(\Psi_0)_z = 2{,}25\text{y}, \tag{29}$$

$$J_0 = \Phi_0(\Omega_0)_x + \Psi_0(\Omega_0)_y + \Omega_0(\Omega_0)_z = 2{,}25z. \tag{30}$$

Equations (28), (29), and (30) are substituted into equation (27), thus obtaining Φ_1, Ψ_1, and Ω_1 as given below, hence

$$\Phi_1 = S^{-1}\big[\theta S[0 - 2{,}25x]\big] = S^{-1}\big[\theta S[-2{,}25x]\big] = S^{-1}[-2{,}25x\theta] = -2{,}25xt,$$

$$\Psi_1 = S^{-1}\big[\theta S[0 - 2{,}25y]\big] = S^{-1}\big[\theta S[-2{,}25y]\big] = S^{-1}[-2{,}25y\theta] = -2{,}25yt,$$

$$\Omega_1 = S^{-1}\big[\theta S[0 - 2{,}25z]\big] = S^{-1}\big[\theta S[-2{,}25z]\big] = S^{-1}[-2{,}25z\theta] = -2{,}25zt.$$

Thus, it is obtained

$$\begin{cases} \Phi_1 = -2{,}25xt, \\ \Psi_1 = -2{,}25yt, \\ \Omega_1 = -2{,}25zt. \end{cases}$$

Then, the p^2 term results in equation (31) as follows

$$\begin{cases} \Phi_2 = S^{-1}\left[\theta S\big[(\Phi_1)_{xx} + (\Phi_1)_{yy} + (\Phi_1)_{zz} - H_1\big]\right], \\ \Psi_2 = S^{-1}\left[\theta S\big[(\Psi_1)_{xx} + (\Psi_1)_{yy} + (\Psi_1)_{zz} - I_1\big]\right], \\ \Omega_2 = S^{-1}\left[\theta S\big[(\Omega_1)_{xx} + (\Omega_1)_{yy} + (\Omega_1)_{zz} - J_1\big]\right]. \end{cases} \tag{31}$$

The values of H_1, I_1, and J_1 are shown in equations (32), (33), and (34).

$$\begin{aligned} H_1 &= \Phi_0(\Phi_1)_x + \Phi_1(\Phi_0)_x + \Psi_0(\Phi_1)_y + \Psi_1(\Phi_0)_y + \Omega_0(\Phi_1)_z + \Omega_1(\Phi_0)_z \\ &= 2{,}5xt - 4{,}5yt - 4{,}5zt, \end{aligned} \tag{32}$$

$$\begin{aligned} I_1 &= \Phi_0(\Psi_1)_x + \Phi_1(\Psi_0)_x + \Psi_0(\Psi_1)_y + \Psi_1(\Psi_0)_y + \Omega_0(\Psi_1)_z + \Omega_1(\Psi_0)_z \\ &= -4{,}5xt + 2{,}5yt - 4{,}5zt, \end{aligned} \tag{33}$$

$$\begin{aligned} J_1 &= \Phi_0(\Omega_1)_x + \Phi_1(\Omega_0)_x + \Psi_0(\Omega_1)_y + \Psi_1(\Omega_0)_y + \Omega_0(\Omega_1)_z + \Omega_1(\Omega_0)_z \\ &= -4{,}5xt - 4{,}5yt + 2{,}5zt. \end{aligned} \tag{34}$$

By substituting equations (32), (33), and (34) into equation (31), we obtain Φ_2, Ψ_2, and Ω_2 as follows

$$\Phi_2 = S^{-1}\big[\theta S[0 - (2{,}5xt - 4{,}5yt - 4{,}5zt)]\big]$$

https://dx.doi.org/10.26594/jmpm.v10i1.5450

$$= S^{-1}\left[\theta S[(-2{,}5x + 4{,}5y + 4{,}5z)t]\right]$$
$$= S^{-1}[(-2{,}5x + 4{,}5y + 4{,}5z)\theta^2]$$
$$= (-2{,}5x + 4{,}5y + 4{,}5z)\frac{t^2}{2}$$
$$= -1{,}125xt^2 + 2{,}25yt^2 + 2{,}25zt^2,$$

$$\Psi_2 = S^{-1}\left[\theta S[0 - (-4{,}5xt + 2{,}5yt - 4{,}5zt)]\right]$$
$$= S^{-1}\left[\theta S[(4{,}5x - 2{,}5y + 4{,}5z)t]\right]$$
$$= S^{-1}[(4{,}5x - 2{,}5y + 4{,}5z)\theta^2]$$
$$= (4{,}5x - 2{,}5yt + 4{,}5z)\frac{t^2}{2}$$
$$= 2{,}25xt^2 - 1{,}125yt^2 + 2{,}25zt^2,$$

$$\Omega_2 = S^{-1}\left[\theta S[0 - (-4{,}5xt - 4{,}5yt + 2{,}5zt)]\right]$$
$$= S^{-1}\left[\theta S[(4{,}5x + 4{,}5y - 2{,}5z)t]\right]$$
$$= S^{-1}[(4{,}5x + 4{,}5y - 2{,}5z)\theta^2]$$
$$= (4{,}5x + 4{,}5y - 2{,}5z)\frac{t^2}{2}$$
$$= 2{,}25xt^2 + 2{,}25yt^2 - 1{,}125zt^2,$$

Thus, it is obtained

$$\begin{cases} \Phi_2 = -1{,}125xt^2 + 2{,}25yt^2 + 2{,}25zt^2, \\ \Psi_2 = 2{,}25xt^2 - 1{,}125yt^2 + 2{,}25zt^2, \\ \Omega_2 = 2{,}25xt^2 + 2{,}25yt^2 - 1{,}125zt^2. \end{cases}$$

The p^3 term results in equation (35) as follows

$$\begin{aligned} \Phi_3 &= S^{-1}\left[\theta S\left[(\Phi_2)_{xx} + (\Phi_2)_{yy} + (\Phi_2)_{zz} - H_2\right]\right], \\ \Psi_3 &= S^{-1}\left[\theta S\left[(\Psi_2)_{xx} + (\Psi_2)_{yy} + (\Psi_2)_{zz} - I_2\right]\right], \\ \Omega_3 &= S^{-1}\left[\theta S\left[(\Omega_2)_{xx} + (\Omega_2)_{yy} + (\Omega_2)_{zz} - J_2\right]\right]. \end{aligned} \tag{35}$$

H_2, I_2, and J_2 are determined by equations (36), (37), and (38).

$$\begin{aligned} H_2 &= \Phi_0(\Phi_2)_x + \Phi_1(\Phi_1)_x + \Phi_2(\Phi_0)_x + \Psi_0(\Phi_2)_y + \Psi_1(\Phi_1)_y + \Psi_2(\Phi_0)_y \\ &\quad + \Omega_0(\Phi_2)_z + \Omega_1(\Phi_1)_z + \Omega_2(\Phi_0)_z \\ &= 15{,}1875xt^2, \end{aligned} \tag{36}$$

$$\begin{aligned} I_2 &= \Phi_0(\Psi_2)_x + \Phi_1(\Psi_1)_x + \Phi_2(\Psi_0)_x + \Psi_0(\Psi_2)_y + \Psi_1(\Psi_1)_y + \Psi_2(\Psi_0)_y \\ &\quad + \Omega_0(\Psi_2)_z + \Omega_1(\Psi_1)_z + \Omega_2(\Psi_0)_z \\ &= 15{,}1875yt^2, \end{aligned} \tag{37}$$

$$\begin{aligned} J_2 &= \Phi_0(\Omega_2)_x + \Phi_1(\Omega_1)_x + \Phi_2(\Omega_0)_x + \Psi_0(\Omega_2)_y + \Psi_1(\Omega_1)_y + \Psi_2(\Omega_0)_y \\ &\quad + \Omega_0(\Omega_2)_z + \Omega_1(\Omega_1)_z + \Omega_2(\Omega_0)_z \\ &= 15{,}1875zt^2. \end{aligned} \tag{38}$$

Equation (36), (37), and (38) are substituted into equation (35), so that

$$
\begin{aligned}
\Phi_3 &= S^{-1}\left[\theta S[0 - 15{,}1875xt^2]\right] \\
&= S^{-1}\left[\theta S[-15{,}1875xt^2]\right] \\
&= S^{-1}[\theta(-15{,}1875x)(2!\,\theta^2)] \\
&= S^{-1}[-30{,}375x\theta^3] \\
&= (-30{,}375x)\frac{t^3}{3!} \\
&= -5{,}0625xt^3,
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3 &= S^{-1}\left[\theta S[0 - 15{,}1875yt^2]\right] \\
&= S^{-1}\left[\theta S[-15{,}1875yt^2]\right] \\
&= S^{-1}[\theta(-15{,}1875y)(2!\,\theta^2)] \\
&= S^{-1}[-30{,}375y\theta^3] \\
&= (-30{,}375y)\frac{t^3}{3!} \\
&= -5{,}0625yt^3,
\end{aligned}
$$

$$
\begin{aligned}
\Omega_3 &= S^{-1}\left[\theta S[0 - 15{,}1875zt^2]\right] \\
&= S^{-1}\left[\theta S[-15{,}1875zt^2]\right] \\
&= S^{-1}[\theta(-15{,}1875z)(2!\,\theta^2)] \\
&= S^{-1}[-30{,}375z\theta^3] \\
&= (-30{,}375z)\frac{t^3}{3!} \\
&= -5{,}0625zt^3.
\end{aligned}
$$

Obtained,

$$
\begin{cases}
\Phi_3 = -5{,}0625xt^3, \\
\Psi_3 = -5{,}0625yt^3, \\
\Omega_3 = -5{,}0625zt^3.
\end{cases}
$$

Next, the p^4 term is shown in equation (39).

$$
\begin{cases}
\Phi_4 = S^{-1}\left[\theta S\left[(\Phi_3)_{xx} + (\Phi_3)_{yy} + (\Phi_3)_{zz} - H_3\right]\right], \\
\Psi_4 = S^{-1}\left[\theta S\left[(\Psi_3)_{xx} + (\Psi_3)_{yy} + (\Psi_3)_{zz} - I_3\right]\right], \\
\Omega_4 = S^{-1}\left[\theta S\left[(\Omega_3)_{xx} + (\Omega_3)_{yy} + (\Omega_3)_{zz} - J_3\right]\right].
\end{cases} \tag{39}
$$

The values of H_3, I_3, and J_3 are obtained as shown in equations (40), (41), and (42).

$$
\begin{aligned}
H_3 &= \Phi_0(\Phi_3)_x + \Phi_1(\Phi_2)_x + \Phi_2(\Phi_1)_x + \Phi_3(\Phi_0)_x + \Psi_0(\Phi_3)_y + \Psi_1(\Phi_2)_y \\
&\quad + \Psi_2(\Phi_1)_y + \Psi_3(\Phi_0)_y + \Omega_0(\Phi_3)_z + \Omega_1(\Phi_2)_z + \Omega_2(\Phi_1)_z + \Omega_3(\Phi_0)_z \\
&= 10{,}125xt^3 - 20{,}25yt^3 - 20{,}25zt^3,
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
I_3 &= \Phi_0(\Psi_3)_x + \Phi_1(\Psi_2)_x + \Phi_2(\Psi_1)_x + \Phi_3(\Psi_0)_x + \Psi_0(\Psi_3)_y + \Psi_1(\Psi_2)_y \\
&\quad + \Psi_2(\Psi_1)_y + \Psi_3(\Psi_0)_y + \Omega_0(\Psi_3)_z + \Omega_1(\Psi_2)_z + \Omega_2(\Psi_1)_z + \Omega_3(\Psi_0)_z
\end{aligned}
$$

$$= -20{,}25xt^3 + 10{,}125yt^3 - 20{,}25zt^3, \quad (41)$$

$$J_3 = \Phi_0(\Omega_3)_x + \Phi_1(\Omega_2)_x + \Phi_2(\Omega_1)_x + \Phi_3(\Omega_0)_x + \Psi_0(\Omega_3)_y + \Psi_1(\Omega_2)_y + \Psi_2(\Omega_1)_y + \Psi_3(\Omega_0)_y + \Omega_0(\Omega_3)_z + \Omega_1(\Omega_2)_z + \Omega_2(\Omega_1)_z + \Omega_3(\Omega_0)_z$$
$$= -20{,}25xt^3 - 20{,}25yt^3 + 10{,}125zt^3. \quad (42)$$

By substituting equations (40), (41), and (42) into equation (39), we obtain

$$\begin{aligned}\Phi_4 &= S^{-1}\big[\theta S[0 - (10{,}125xt^3 - 20{,}25yt^3 - 20{,}25zt^3)]\big]\\ &= S^{-1}\big[\theta S[(-10{,}125x + 20{,}25y + 20{,}25z)t^3]\big]\\ &= S^{-1}[\theta(-10{,}125x + 20{,}25y + 20{,}25z)(3!\,\theta^3)]\\ &= S^{-1}[(-60{,}75x + 121{,}5y + 121{,}5z)\theta^4]\\ &= (-60{,}75x + 121{,}5y + 121{,}5z)\frac{t^4}{4!}\\ &= -2{,}53125xt^4 + 5{,}0625yt^4 + 5{,}0625zt^4,\end{aligned}$$

$$\begin{aligned}\Psi_4 &= S^{-1}\big[\theta S[0 - (-20{,}25xt^3 + 10{,}125yt^3 - 20{,}25zt^3)]\big]\\ &= S^{-1}\big[\theta S[(20{,}25x - 10{,}125y + 20{,}25z)t^3]\big]\\ &= S^{-1}[\theta(20{,}25x - 10{,}125y + 20{,}25z)(3!\,\theta^3)]\\ &= S^{-1}[(121{,}5x - 60{,}75y + 121{,}5z)\theta^4]\\ &= (121{,}5x - 60{,}75y + 121{,}5z)\frac{t^4}{4!}\\ &= 5{,}0625xt^4 - 2{,}53125yt^4 + 5{,}0625zt^4,\end{aligned}$$

$$\begin{aligned}\Omega_4 &= S^{-1}\big[\theta S[0 - (-20{,}25xt^3 - 20{,}25yt^3 + 10{,}125zt^3)]\big]\\ &= S^{-1}\big[\theta S[(20{,}25x + 20{,}25y - 10{,}125z)t^3]\big]\\ &= S^{-1}[\theta(20{,}25x + 20{,}25y - 10{,}125z)(3!\,\theta^3)]\\ &= S^{-1}[(121{,}5x + 121{,}5y - 60{,}75z)\theta^4]\\ &= (121{,}5x + 121{,}5y - 60{,}75z)\frac{t^4}{4!}\\ &= 5{,}0625xt^4 + 5{,}0625yt^4 - 2{,}53125zt^4.\end{aligned}$$

Thus, it is obtained

$$\begin{cases}\Phi_4 = -2{,}53125xt^4 + 5{,}0625yt^4 + 5{,}0625zt^4,\\ \Psi_4 = 5{,}0625xt^4 - 2{,}53125yt^4 + 5{,}0625zt^4,\\ \Omega_4 = 5{,}0625xt^4 + 5{,}0625yt^4 - 2{,}53125zt^4.\end{cases}$$

Next, the p^5 term is given by equation (43).

$$\begin{cases}\Phi_5 = S^{-1}\Big[\theta S\big[(\Phi_4)_{xx} + (\Phi_4)_{yy} + (\Phi_4)_{zz} - H_4\big]\Big],\\ \Psi_5 = S^{-1}\Big[\theta S\big[(\Psi_4)_{xx} + (\Psi_4)_{yy} + (\Psi_4)_{zz} - I_4\big]\Big],\\ \Omega_5 = S^{-1}\Big[\theta S\big[(\Omega_4)_{xx} + (\Omega_4)_{yy} + (\Omega_4)_{zz} - J_4\big]\Big].\end{cases} \quad (43)$$

The values of H_4, I_4, and J_4 are presented in equations (44), (45), and (46).

$$
\begin{aligned}
H_4 &= \Phi_0(\Phi_4)_x + \Phi_1(\Phi_3)_x + \Phi_2(\Phi_2)_x + \Phi_3(\Phi_1)_x + \Phi_4(\Phi_0)_x + \Psi_0(\Phi_4)_y \\
&\quad +\Psi_1(\Phi_3)_y + \Psi_2(\Phi_2)_y + \Psi_3(\Phi_1)_y + \Psi_4(\Phi_0)_y + \Omega_0(\Phi_4)_z + \Omega_1(\Phi_3)_z \\
&\quad +\Omega_2(\Phi_2)_z + \Omega_3(\Phi_1)_z + \Omega_4(\Phi_0)_z \\
&= 56{,}953125xt^4, \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= \Phi_0(\Psi_4)_x + \Phi_1(\Psi_3)_x + \Phi_2(\Psi_2)_x + \Phi_3(\Psi_1)_x + \Phi_4(\Psi_0)_x + \Psi_0(\Psi_4)_y \\
&\quad +\Psi_1(\Psi_3)_y + \Psi_2(\Psi_2)_y + \Psi_3(\Psi_1)_y + \Psi_4(\Psi_0)_y + \Omega_0(\Psi_4)_z + \Omega_1(\Psi_3)_z \\
&\quad +\Omega_2(\Psi_2)_z + \Omega_3(\Psi_1)_z + \Omega_4(\Psi_0)_z \\
&= 56{,}953125yt^4, \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
J_4 &= \Phi_0(\Omega_4)_x + \Phi_1(\Omega_3)_x + \Phi_2(\Omega_2)_x + \Phi_3(\Omega_1)_x + \Phi_4(\Omega_0)_x + \Psi_0(\Omega_4)_y \\
&\quad +\Psi_1(\Omega_3)_z + \Psi_2(\Omega_2)_y + \Psi_3(\Omega_1)_y + \Psi_4(\Omega_0)_y + \Omega_0(\Omega_4)_z + \Omega_1(\Omega_3)_z \\
&\quad +\Omega_2(\Omega_2)_z + \Omega_3(\Omega_1)_z + \Omega_4(\Omega_0)_z \\
&= 56{,}953125zt^4. \qquad (46)
\end{aligned}
$$

Equations (44), (45), and (46) are substituted into equation (43), so that

$$
\begin{aligned}
\Phi_5 &= S^{-1}\left[\theta S[0 - 56{,}953125xt^4]\right] \\
&= S^{-1}\left[\theta S[-56{,}953125xt^4]\right] \\
&= S^{-1}[\theta(-56{,}953125x)(4!\,\theta^4)] \\
&= S^{-1}[-1366{,}875x\theta^5] \\
&= (-1366{,}875x)\frac{t^5}{5!} \\
&= -11{,}390625xt^5,
\end{aligned}
$$

$$
\begin{aligned}
\Psi_5 &= S^{-1}\left[\theta S[0 - 56{,}953125yt^4]\right] \\
&= S^{-1}\left[\theta S[-56{,}953125yt^4]\right] \\
&= S^{-1}[\theta(-56{,}953125y)(4!\,\theta^4)] \\
&= S^{-1}[-1366{,}875y\theta^5] \\
&= (-1366{,}875y)\frac{t^5}{5!} \\
&= -11{,}390625yt^5,
\end{aligned}
$$

$$
\begin{aligned}
\Omega_5 &= S^{-1}\left[\theta S[0 - 56{,}953125zt^4]\right] \\
&= S^{-1}\left[\theta S[-56{,}953125zt^4]\right] \\
&= S^{-1}[\theta(-56{,}953125z)(4!\,\theta^4)] \\
&= S^{-1}[-1366{,}875z\theta^5] \\
&= (-1366{,}875z)\frac{t^5}{5!} \\
&= -11{,}390625zt^5.
\end{aligned}
$$

Thus, it is obtained

$$
\begin{cases}
\Phi_5 = -11{,}390625xt^5, \\
\Psi_5 = -11{,}390625yt^5, \\
\Omega_5 = -11{,}390625zt^5.
\end{cases}
$$

Thus, the solutions of the series Φ, Ψ, and Ω are as follows

$$\begin{aligned}\Phi &= \Phi_0 + p\Phi_1 + p^2\Phi_2 + p^3\Phi_3 + p^4\Phi_4 + p^5\Phi_5 + \cdots \\ &= (-0{,}5x + y + z) + p(-2{,}25xt) + p^2(-1{,}125xt^2 + 2{,}25yt^2 + 2{,}25zt^2) \\ &\quad + p^3(-5{,}0625xt^3) + p^4(-2{,}53125xt^4 + 5{,}0625yt^4 + 5{,}0625zt^4) \\ &\quad + p^5(-11{,}390625xt^5) + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi &= \Psi_0 + p\Psi_1 + p^2\Psi_2 + p^3\Psi_3 + p^4\Psi_4 + p^5\Psi_5 + \cdots \\ &= (x - 0{,}5y + z) + p(-2{,}25yt) + p^2(2{,}25xt^2 - 1{,}125yt^2 + 2{,}25zt^2) \\ &\quad + p^3(-5{,}0625yt^3) + p^4(5{,}0625xt^4 - 2{,}53125yt^4 + 5{,}0625zt^4) \\ &\quad + p^5(-11{,}390625yt^5) + \cdots,\end{aligned}$$

$$\begin{aligned}\Omega &= \Omega_0 + p\Omega_1 + p^2\Omega_2 + p^3\Omega_3 + p^4\Omega_4 + p^5\Omega_5 + \cdots \\ &= (x + y - 0{,}5z) + p(-2{,}25zt) + p^2(2{,}25xt^2 + 2{,}25yt^2 - 1{,}125zt^2) \\ &\quad + p^3(-5{,}0625zt^3) + p^4(5{,}0625xt^4 + 5{,}0625yt^4 - 2{,}53125zt^4) \\ &\quad + p^5(-11{,}390625zt^5) + \cdots,\end{aligned}$$

Furthermore, the approximation solutions ϕ, ψ, and ω are

$$\begin{aligned}\phi(x,y,z,t) &= \lim_{p\to 1}\Phi \\ &= -0{,}5x + y + z - 2{,}25xt - 1{,}125xt^2 + 2{,}25yt^2 + 2{,}25zt^2 - 5{,}0625xt^3 \\ &\quad - 2{,}53125xt^4 + 5{,}0625yt^4 + 5{,}0625zt^4 - 11{,}390625xt^5 + \cdots,\end{aligned}$$

$$\begin{aligned}\psi(x,y,z,t) &= \lim_{p\to 1}\Psi \\ &= x - 0{,}5y + z - 2{,}25yt + 2{,}25xt^2 - 1{,}125yt^2 + 2{,}25zt^2 - 5{,}0625yt^3 \\ &\quad + 5{,}0625xt^4 - 2{,}53125yt^4 + 5{,}0625zt^4 - 11{,}390625yt^5 + \cdots,\end{aligned}$$

$$\begin{aligned}\omega(x,y,z,t) &= \lim_{p\to 1}\Omega \\ &= x + y - 0{,}5z - 2{,}25zt + 2{,}25xt^2 + 2{,}25yt^2 - 1{,}125zt^2 - 5{,}0625zt^3 \\ &\quad + 5{,}0625xt^4 + 5{,}0625yt^4 - 2{,}53125zt^4 - 11{,}390625zt^5 + \cdots.\end{aligned}$$

Thus, it is obtained

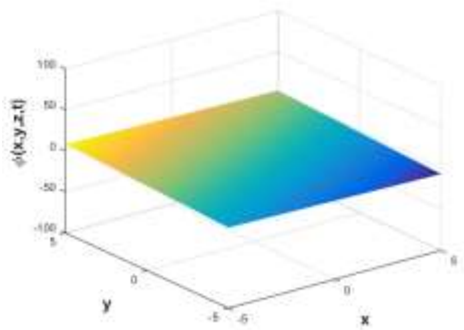
$$\begin{cases}\phi(x,y,z,t) = \big((-0{,}5x + y + z) - 2{,}25xt\big)(1 + 2{,}25t^2 + 5{,}0625t^4 + \cdots), \\ \psi(x,y,z,t) = \big((x - 0{,}5y + z) - 2{,}25yt\big)(1 + 2{,}25t^2 + 5{,}0625t^4 + \cdots), \\ \omega(x,y,z,t) = \big((x + y - 0{,}5z) - 2{,}25zt\big)(1 + 2{,}25t^2 + 5{,}0625t^4 + \cdots).\end{cases}$$

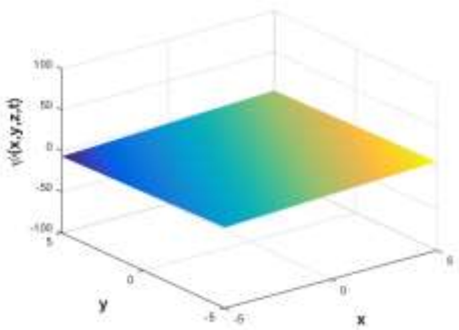
Given that $(1 + 2{,}25t^2 + 5{,}0625t^4 + \cdots)$ is the Maclaurin series of the function $f(t) = (1 - 2{,}25t^2)^{-1}$, we get

$$\begin{cases}\phi(x,y,z,t) = \frac{(-0{,}5x+y+z)-2{,}25xt}{1-2{,}25t^2}, \text{with the condition } 2{,}25t^2 \neq 1, \\ \psi(x,y,z,t) = \frac{(x-0{,}5y+z)-2{,}25yt}{1-2{,}25t^2}, \quad \text{with the condition } 2{,}25t^2 \neq 1, \\ \omega(x,y,z,t) = \frac{(x+y-0{,}5z)-2{,}25zt}{1-2{,}25t^2}, \quad \text{with the condition } 2{,}25t^2 \neq 1.\end{cases} \tag{47}$$

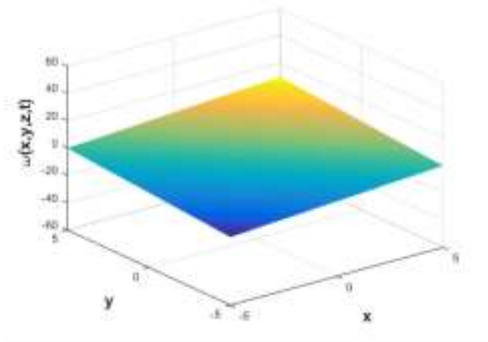
https://dx.doi.org/10.26594/jmpm.v10i1.5450

Based on equation (47), the solution of the 3D Burgers equation (1) using HPM-ST is equivalent to the exact solution of the 3D Burgers equation shown in equation (3). The solution obtained using the HPM-ST method also corresponds to the solution of the 3D Burgers equation (1) obtained using the Laplace VIM, as presented in the research by Singh and Singh (2022). The visualization of the solution of the 3D Burgers equation (1) at $t = 0$ can be seen in Figure 1 and at $t = 0{,}6$ can be seen in Figure 2.



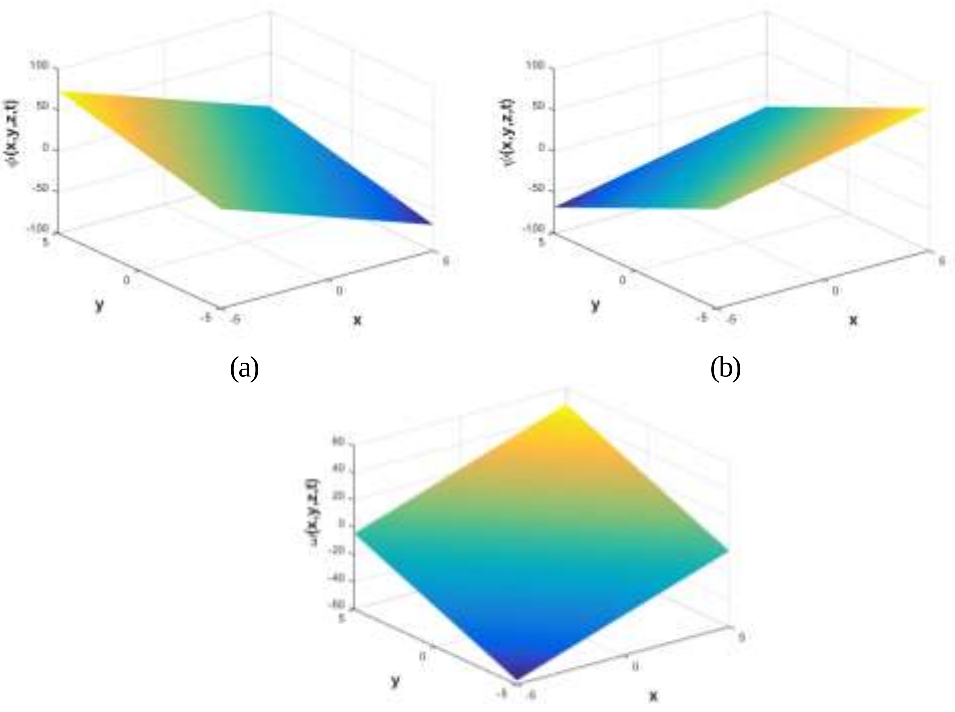


(a) (b)



(c)

Figure 1. The visualization of the solution of 3D Burgers equation (a) $\phi(x, y, z, t)$ at $t = 0$ (b) $\psi(x, y, z, t)$ at $t = 0$ (c) $\omega(x, y, z, t)$ at $t = 0$



(a) (b)

(c)

Figure 2. The visualization of the solution of 3D Burgers equation (a) $\phi(x,y,z,t)$ at $t = 0,6$ (b) $\psi(x,y,z,t)$ at $t = 0,6$ (c) $\omega(x,y,z,t)$ at $t = 0,6$

CONCLUSION AND SUGGESTIONS

Based on the results and discussion presented, it can be concluded that HPM-ST has been successfully applied to the 3D Burgers equation. The solution obtained is also equivalent to the exact solution. The solution of the 3D Burgers equation in equation (1) with initial value given in equation (2) using HPM-ST is:

$$\begin{cases} \phi(x,y,z,t) = \dfrac{(-0{,}5x + y + z) - 2{,}25xt}{1 - 2{,}25t^2}, \text{with the condition } 2{,}25t^2 \neq 1, \\ \psi(x,y,z,t) = \dfrac{(x - 0{,}5y + z) - 2{,}25yt}{1 - 2{,}25t^2}, \quad \text{with the condition } 2{,}25t^2 \neq 1, \\ \omega(x,y,z,t) = \dfrac{(x + y - 0{,}5z) - 2{,}25zt}{1 - 2{,}25t^2}, \quad \text{with the condition } 2{,}25t^2 \neq 1. \end{cases}$$

In practice, more complex equations, such as the fractional Schrödinger-KdV equation, pose a greater challenge in the application of HPM-ST. Therefore, future research can apply HPM-ST to other more complex differential equations or compare HPM-ST with a combination of the Sumudu transform with different analytical methods to solve the 3D Burgers equation.

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