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On the Hardy-Littlewood Maximal Operator in Zorko Spaces

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Abstract

Zorko spaces are introduced to address the density issue in Morrey spaces, which are defined by utilizing the difference of a function of first order. Later the difference of a function of second order is also employed to define modified Zorko spaces and approximation properties are investigated therein. On the other hand, integral operators are mostly studied on the boundedness properties in Morrey spaces and its variants. In this paper, integral operators, especially Hardy-Littlewood maximal operators and fractional Hardy-Littlewood maximal operators, are investigated on its boundedness properties in Zorko spaces. Furthermore, closedness of Zorko space under Hardy-Littlewood maximal operators has been proved.

Keywords: *maximal operator; Hardy-Littlewood; boundedness; closedness; Zorko spaces*

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INTRODUCTION

Morrey spaces as a generalization of Lebesgue spaces have different properties in terms of the density of smooth functions than those of Lebesgue spaces. While the set of smooth functions is dense in Lebesgue spaces, Zorko (1986) provided an example of function in Morrey spaces which failed to be approximated by smooth functions or even continuous ones. To address the issue, Zorko defined a subspace of $L^{p,\lambda}$, which is then called Zorko space, whose translation of each member is continuous, that is $\|f(\cdot - y) - f\| \to 0$ as $|y| \to 0$. In the paper, Zorko proved that the subspace is the closure of the functions in $C^\infty(\mathbb{R}^n)$ in the form of a convolution with a mollifier. Following this result, Hasanah & Gunawan (2024) modified the Zorko space by employing the second order of difference and proved that the space is a closure of smooth functions in Morrey space, and similar result was obtained in the variant of Morrey space in Hasanah & Gunawan (2025).

Having established the structural properties of Zorko spaces, attention naturally turns to the kinds of operators that act on the spaces. Because infinite-dimensional spaces admit many unbounded operators, determining boundedness becomes crucial in understanding the analytic behavior these spaces can support. In this context, integral operators, such as maximal operators, serve as an important class to investigate, as they appear widely in PDEs and potential theory. The study of maximal operators is one of the most important topics in harmonic analysis (Gogatishvili & Mustafayev, 2015). Analyzing the boundedness of an operator within normed spaces is equivalent to establishing its continuity or the operators which allows the use of powerful theorems and tools from harmonic analysis. This continuity is crucial for practical applications, theoretical development in functional analysis, and modeling physical phenomena.

Morrey spaces, in certain instances, can accommodate the boundedness properties of operators that cannot be addressed within the framework of Lebesgue spaces. The boundedness of the Hardy-Littlewood maximal operators was established by Chiarenza & Frasca (1987), while Adams & Xiao (2004) investigated the fractional integral operator and the fractional maximal operator under the equivalence of the Morrey norms. The study of the boundedness of singular operators including the Hardy-Littlewood maximal operator and the fractional Hardy-Littlewood maximal operator has been conducted within various extensions of Morrey spaces, such as Ho (2013), Zhou & Zhao (2022), Ramadana & Gunawan (2023), and Hao et al. (2024). This paper examines the classical singular operators, namely the Hardy-Littlewood maximal operator and the fractional Hardy-Littlewood maximal operator on Morrey spaces, with a focus on Zorko spaces.

Although several works have addressed the structural properties of Morrey-type spaces, the analysis of how maximal operators interact with the internal stability of Zorko spaces represents a natural and significant direction alongside the study of boundedness. This perspective aligns with the work of Deringoz (2024), who demonstrated that the vanishing property defining the vanishing Orlicz–Morrey space is preserved under maximal operators, as well as Diarra & Fofana (2023), who explored invariance phenomena within certain families of Banach spaces under the Hardy–Littlewood maximal operator. However, despite those contributions, the literature has not yet provided whether the closedness property in terms of translation invariance is preserved within the framework of Zorko spaces. To complement the boundedness results, this paper therefore investigates the invariance properties of Zorko spaces under the action of the Hardy–Littlewood maximal operator and the fractional Hardy–Littlewood maximal operator with a focus on establishing translation-invariant structure of the space.

Preliminaries

Morrey spaces introduced by Morrey (1938) play a pivotal role in the study of elliptical partial differential equations. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$, $1 \leq p < \infty$, $0 \leq \lambda \leq n$ consists of all p-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\lambda}} := \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda}\int_{B(x,r)} |f(y)|^p \, dy\right)^{\frac{1}{p}} < \infty.$$

Throughout this paper, $B(x,r)$ denotes an open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. For a set E in $\mathbb{R}^n$, if E is measurable set in $\mathbb{R}^n$, then $|E|$ denotes the Lebesgue measure of E. Morrey spaces can be considered as a generalization of Lebesgue spaces. However, the density property of smooth functions that holds in Lebesgue spaces does not extend to Morrey spaces. Zorko (1986) proposed a closed subset of a Morrey space in which the translation of each function is continuous and proved the denseness property in the subset which later is called Zorko space.

The Zorko space $\mathbb{L}^{p,\lambda}(\mathbb{R}^n)$ consists of all functions $f \in L^{p,\lambda}(\mathbb{R}^n)$ which have the property of continuous translation, i.e.,

$$\mathbb{L}^{p,\lambda}(\mathbb{R}^n) := \left\{f \in L^{p,\lambda} : \|f(x-y) - f(x)\|_{\mathcal{M}^{p,\lambda}} \to 0 \text{ as } |y| \to 0\right\},$$

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For $0 < \lambda < n$, we have an inclusion $\mathbb{L}^{p,\lambda}(\mathbb{R}^n) \subset V_0 L^{p,\lambda}(\mathbb{R}^n) \subset L^{p,\lambda}(\mathbb{R}^n)$, where $V_0 L^{p,\lambda}(\mathbb{R}^n)$ denotes the vanishing Morrey subspace at the origin. Taking $\lambda = 0$ results in the relation $\mathbb{L}^{p,0}(\mathbb{R}^n) = V_0 L^{p,0}(\mathbb{R}^n) = L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. The Zorko space is a closed sunspace in a Morrey space. The density issue of smooth functions in Morrey spaces is dealt with using mollifiers. From now on, we omit the reference of the domain when it is the whole $\mathbb{R}^n$.

The Hardy-Littlewood maximal operator M dan fractional maximal operator M_α, where $0 < \alpha < n$, are defined by

$$Mf(x) \coloneqq \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z)|\, dz \,, x \in \mathbb{R}^n,$$

and

$$M_\alpha f(x) \coloneqq \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int_{B(x,r)} |f(z)|\, dz \,, x \in \mathbb{R}^n,$$

for locally integrable functions f on $\mathbb{R}^n$. It is well known that M is bounded on Lebesgue spaces L^p for $1 < p \leq \infty$ and on the general Lebesgue space $L^{p(x)}$ as shown in (Nekvinda, 2004). The boundedness property of the Hardy-Littlewood maximal operator in Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ has been proved in (Adams, 2015) as stated in the following.

Theorem 1. For any function f in Morrey space $L^{p,\lambda}$ with $1 < p < \infty$, $0 < \lambda \leq n$, the Hardy-Littlewood maximal operator M is bounded in $L^{p,\lambda}$, i.e. there is a constant C independent of f such that

$$\|Mf\|_{L^{p,\lambda}} \leq C\, \|f\|_{L^{p,\lambda}}.$$

This boundedness property of maximal operators ensures that tools like the Calderón-Zygmund decomposition or Sobolev-type embeddings can be adapted to the Morrey setting.

Boundedness on Morrey spaces often serves as a bridge between different scales of function spaces. The fractional maximal operator M_α with $0 < \alpha < n$, maps $L^{p,\lambda}$ boundedly into $L^{p,\mu}$ stated in the following theorem.

Theorem 2. Let f be a function in Morrey space $L^{p,\lambda}$, $1 < p < \frac{n}{\alpha}$, $0 \leq \lambda < n - \alpha p$, and define

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}, \mu = \frac{q}{p}\lambda.$$

The Hardy-Littlewood fractional maximal operator M_α is bounded from $L^{p,\lambda}$ into $L^{q,\mu}$ under the relation

$$\|M_\alpha f\|_{L^{q,\mu}} \leq C\, \|f\|_{L^{p,\lambda}},$$

where C depends only on n, p, λ, α.

METHODS

The research employs a theoretical analytical approach to investigate the boundedness of maximal operators in Zorko spaces and to establish their closedness properties. The study begins with a comprehensive review of the fundamental concepts of Morrey and Zorko spaces, followed by definition-based analysis on Hardy-Littlewood maximal operators and its properties on Morrey spaces. The boundedness of the Hardy–Littlewood maximal operator and its fractional counterpart in Zorko space is then formulated using the definition of Zorko space and corresponding boundedness theorems of the operators in Morrey spaces. Employing the property of translation preservation by maximal operators ensures the next step, i.e. preservation of structural information in Zorko space. The methodology relies on functional analytic techniques, norm equivalence arguments, and existing results from Morrey spaces to ensure the validity and generality of the findings.

RESULTS AND DISCUSSION

The boundedness properties of the Hardy–Littlewood maximal operator and the fractional Hardy–Littlewood maximal operator within Morrey spaces have been extensively examined in prior works, including those by Nekvinda (2004), Adams & Xiao (2012), Adams (2015), and (Gunawan & Schwanke, 2019). Building upon these foundational studies, this section focuses on functions belonging to Zorko spaces. Since a Zorko space constitutes a subspace of a Morrey space, any function $f$residing in a Zorko space inherits the boundedness property of its image under the action of the Hardy-Littlewood maximal operator or its fractional analogue, as established by Theorem 1 and Theorem 2. Specifically, the detailed formulation of this boundedness property is presented as follows.

Theorem 3. Let $f \in \mathbb{L}^{p,\lambda}$, $1 < p < \infty$, and $0 < \lambda, \alpha < n$. The Hardy-Littlewood maximal operator M and the fractional Hardy-Littlewood maximal operator M_α act on f are bounded, i.e. there is a constant C independent from f such that

$$\|Mf\|_{L^{p,\lambda}} \leq C\ \|f\|_{L^{p,\lambda}}$$

and

$$\|M_\alpha f\|_{L^{p,\lambda}} \leq C\ \|f\|_{L^{q,\mu}}.$$

In the next discussion, we turn our attention to the closedness property of Zorko spaces under the action of maximal operators. While the boundedness of these operators on Morrey and Zorko spaces has been established, the question of whether the image of a function under such operators remains within the same Zorko space has not been fully addressed in earlier studies. Specifically, for a function $f \in L^{p,\lambda}$ belonging to a Zorko space, it remains an open problem whether the image Mfor $M_\alpha f$, produced by the Hardy-Littlewood maximal operator or its fractional counterpart, also lies in the Zorko space. Investigating this property is essential for understanding the structural stability of Zorko spaces under these operators, as it provides insight into whether these spaces are invariant or closed with respect to the application of maximal and fractional maximal operators.

The Zorko space can be regarded as a subspace of a Morrey space that consists of functions exhibiting continuity with respect to translation. In other words, for any function f belonging to a Zorko space, the norm of the translated function $f(\cdot + y)$ approaches that of f as the translation parameter $|y|$ tends to zero, ensuring a form of smoothness and stability under small shifts. This translation continuity distinguishes Zorko spaces from general Morrey spaces and plays a crucial role in the analysis of operator behavior. The following lemma establishes an important property of maximal operators in this context, namely their invariance under translation. This means that applying a maximal operator to a translated function yields a translation of the operator's image, preserving the functional structure within the Zorko space. Such a property is fundamental for further demonstrating the closedness of Zorko spaces under maximal operators and for understanding how these operators interact with translation-continuous functions.

Lemma 4. Let $f \in \mathbb{L}^{p,\lambda}$, $1 < p < \infty$, and $0 < \lambda, \alpha < n$. Zorko spaces are translation invariant under the Hardy-Littlewood maximal operator and the fractional Hardy-Littlewood maximal operator, that is $M(\tau_y f)(x) = \tau_y(Mf)(x)$, where $\tau_y Mf$ is defined by

$$(\tau_y Mf)(x) \coloneqq (Mf)(x-y), \qquad x, y \in \mathbb{R}^n.$$

Proof. By the definition of the Hardy-Littlewood maximal operator and changing variable we have

$$M(\tau_y f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\tau_y f(z)|\, dz$$

$$= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z-y)|\, dz$$

$$= \sup_{r>0} \frac{1}{|B(x-y,r)|} \int_{B(x-y,r)} |f(u)|\, du$$

$$= (Mf)(x-y)$$

$$= (\tau_y Mf)(x).$$

By similar argument, we also have that $M_\alpha(\tau_y f)(x) = (\tau_y M_\alpha f)(x)$.

The subsequent lemma establishes a relationship between the translation of a function belonging to a Zorko space and the corresponding transformation under the action of maximal operators. Specifically, it characterizes how the behavior of a function under translation is reflected through the operation of the Hardy–Littlewood maximal operator and its fractional counterpart. The result serves as a basic step in analyzing whether Zorko spaces remain invariant or closed under these operators.

Lemma 5. Let $f \in \mathbb{L}^{p,\lambda}$, $1 < p < \infty$, and $0 < \lambda, \alpha < n$. The following inequality holds

$$M(\tau_y f)(x) - Mf(x) \le M(\tau_y f - f)(x), x \in \mathbb{R}^n$$

and

$$M_\alpha(\tau_y f)(x) - M_\alpha f(x) \le M_\alpha(\tau_y f - f)(x), x \in \mathbb{R}^n$$

Proof. Using the definition of the Hardy-Littlewood maximal operator and the triangle inequality obtains

$$M(\tau_y f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\tau_y f(z)|\, dz$$

$$= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z-y)|\, dz$$

$$\leq \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z-y) - f(z)| + |f(z)|\, dz$$

$$\leq \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z-y) - f(z)|\, dz + \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z)|\, dz$$

$$\leq \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |(\tau_y f - f)(z)|\, dz + \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(z)|\, dz$$

$$= M(\tau_y f - f)(x) + Mf(x).$$

Thus, we have

$$M(\tau_y f)(x) - Mf(x) \leq M(\tau_y f - f)(x).$$

This property is also held for fractional counterpart of maximal operator by similar arguments. We are now prepared to establish the closedness property of Zorko spaces under the action of maximal operators.

Theorem 6. Let $f \in \mathbb{L}^{p,\lambda}$, $1 < p < \infty$, and $0 < \lambda, \alpha < n$. The Zorko space $\mathbb{L}^{p,\lambda}$ is closed under the action of Hardy-Littlewood maximal operator and fractional Hardy Littlewood maximal operators, i.e. Mf is in $\mathbb{L}^{p,\lambda}$ and $M_\alpha f$ is in $\mathbb{L}^{q,\mu}$ for

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}, \mu = \frac{q}{p}\lambda.$$

Proof. By virtue of Lemma 4 and Lemma 5, we get

$$Mf(x-y) - Mf(x) = \tau_y Mf(x) - Mf(x)$$

$$= M(\tau_y f)(x) - Mf(x)$$

$$\leq M(\tau_y f - f)(x).$$

Taking Morrey norm for above inequality and by virtue of Theorem 3 have obtained

$$\|Mf(x-y)-Mf(x)\|_{L^{p,\lambda}} \le \left\|M\left(\tau_y f - f\right)(x)\right\|_{L^{p,\lambda}}$$

$$\le C\,\left\|\tau_y f - f\right\|_{L^{p,\lambda}}.$$

Taking limit of both sides, as $|y| \to 0$ we have

$$\lim_{|y|\to 0}\|Mf(x-y)-Mf(x)\|_{L^{p,\lambda}} \le \lim_{|y|\to 0}\left\|\tau_y f - f\right\|_{L^{p,\lambda}} = 0$$

as $f \in \mathbb{L}^{p,\lambda}$. This states that Mf is in Zorko space $\mathbb{L}^{p,\lambda}$. By similar argument, we also have

$$\|M_\alpha f(x-y)-M_\alpha f(x)\|_{L^{q,\mu}} \le C\,\left\|\tau_y f - f\right\|_{L^{p,\lambda}}.$$

Taking limit on both sides ensures the membership of $M_\alpha f$ in $\mathbb{L}^{q,\mu}$.

In previous studies, specifically in Hasanah & Gunawan (2024) and Hasanah & Gunawan (2025), Zorko space has the property that is closed under convolution with integrable kernel. The result in this paper strengthens the stability in terms of translation and the closedness of Zorko spaces.

CONCLUSION AND SUGGESTIONS

The Hardy-Littlewood maximal operator and its fractional variant are bounded on Zorko spaces. Moreover, the result confirms that Zorko spaces preserve translation continuity under the Hardy–Littlewood maximal operator and its fractional counterpart. This finding represents a significant step toward establishing the closedness properties of Zorko spaces within the broader context of Morrey-type function spaces.

A promising avenue for further studies would be to investigate whether other fundamental operators in harmonic analysis, such as Calderón–Zygmund singular integrals, Riesz potentials, or commutators, also act boundedly on Zorko spaces. Additionally, exploring interpolation properties and duality theory for Zorko spaces could further clarify their role as a bridge between classical Morrey spaces and more general function spaces with variable or mixed norms.

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REFERENCES

Adams, D. R. (2015). *Morrey spaces* (J. J. Benedettoo, Ed.). Springer International Publishing.

Adams, D. R., & Xiao, J. (2004). Nonlinear potential analysis on Morrey spaces and their capacities. *Indiana University Mathematics Journal*, *53*(6), 1631–1666. https://doi.org/10.1512/iumj.2004.53.2470

Adams, D. R., & Xiao, J. (2012). Morrey spaces in harmonic analysis. *Arkiv for Matematik*, *50*(2), 201–230. https://doi.org/10.1007/s11512-010-0134-0

Chiarenza, F., & Frasca, M. (1987). Morrey spaces and Hardy-Littlewood maximal function. *Rend. Mat. Apple. (7)*, *7*, 273–279.

Deringoz, F. (2024). Hardy–Littlewood maximal operator in a new vanishing Orlicz–Morrey space. *Zeitschrift Für Analysis Und Ihre Anwendungen*. https://doi.org/10.4171/zaa/1778

Diarra, N., & Fofana, I. (2023). Characterization of some closed linear subspaces of Morrey spaces and approximation. *Advances in Pure and Applied Mathematics*, *14*(3), 41–72. https://doi.org/10.21494/ISTE.OP.2023.0980

Gogatishvili, A., & Mustafayev, R. Ch. (2015). A note on boundedness of the Hardy-Littlewood maximal operator on Morrey spaces. *Mediterranean Journal of Mathematics*, *13*(4), 1885–1891. https://doi.org/10.1007/s00009-015-0614-3

Gunawan, H., & Schwanke, C. (2019). The Hardy–Littlewood maximal operator on discrete morrey spaces. *Mediterranean Journal of Mathematics*, *16*(1). https://doi.org/10.1007/s00009-018-1277-7

Hao, X. B., Li, B. D., & Yang, S. (2024). The Hardy–Littlewood maximal operator on discrete weighted Morrey spaces. *Acta Mathematica Hungarica*, *172*(2), 445–469. https://doi.org/10.1007/s10474-024-01420-3

Hasanah, D., & Gunawan, H. (2024). Approximation in modified Zorko spaces. *Analysis Mathematica*, *50*(2), 553–561. https://doi.org/10.1007/S10476-024-00025-W/METRICS

Hasanah, D., & Gunawan, H. (2025). Approximation in generalized Morrey spaces using the second-order difference. *Mathematical Inequalities & Applications*, *28*(1), 159–172. https://doi.org/10.7153/mia-2025-28-11

Ho, K.-P. (2013). The fractional integral operators on Morrey spaces with variable exponent on unbounded domains. *Mathematical Inequalities & Applications*, (2), 363–373. https://doi.org/10.7153/mia-16-27

Morrey, C. B. (1938). On the solutions of quasi-linear elliptic partial differential equations. *Transactions of the American Mathematical Society*, *43*(1), 126. https://doi.org/10.2307/1989904

Nekvinda, A. (2004). Hardy-Littlewood maximal operator on L^p(x)(R). *Mathematical Inequalities & Applications*, *7*(2), 255–265.

Ramadana, Y., & Gunawan, H. (2023). Boundedness of the Hardy–Littlewood maximal operator, fractional integral operators, and calderón–zygmund operators on generalized weighted morrey spaces. *Khayyam Journal of Mathematics*, *9*(2), 263–287. https://doi.org/10.22034/KJM.2023.386608.2779

Zhou, J., & Zhao, F. (2022). Boundedness of the fractional Hardy-Littlewood maximal operator on weighted Morrey spaces. *Analysis and Mathematical Physics*, *12*(4), 87. https://doi.org/10.1007/s13324-022-00695-5

Zorko, C. T. (1986). Morrey space. *Proceedings of the American Mathematical Society*, *98*(4), 586–592.